

THE QUASI-HOPF ANALOGUE OF $\mathbf{u}_q(\mathfrak{sl}_2)$

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ABSTRACT. In [4], some quasi-Hopf algebras of dimension n^3 , which can be understood as the quasi-Hopf analogues of Taft algebras, are constructed. Moreover, the quasi-Hopf analogues of generalized Taft algebras are considered in [7], where the language of the dual of a quasi-Hopf algebra is used. The Drinfeld doubles of such quasi-Hopf algebras are computed in this paper. The authors in [5] shew that the Drinfeld double of a quasi-Hopf algebra of dimension n^3 constructed in [4] is always twist equivalent to Lusztig's small quantum group $\mathbf{u}_q(\mathfrak{sl}_2)$ if n is odd. Based on computations and analysis, we show that this is *not* the case if n is even. That is, the quasi-Hopf analogue $\mathbf{Q}\mathbf{u}_q(\mathfrak{sl}_2)$ of $\mathbf{u}_q(\mathfrak{sl}_2)$ is gotten.

1. INTRODUCTION

Historically, the Lusztig's definition of a quantum group [9] opens a convenient door for a pure mathematician to go into the field of quantum groups. Do we have such definition for a quasi-quantum group? For a simple finite-dimensional Lie algebra \mathfrak{g} over \mathbb{C} , Drinfeld [Proposition 3.16, 3] told us that a quasitriangular quasi-Hopf quantized enveloping algebra $U\mathfrak{g}[[\hbar]]$ is indeed twist equivalent to the usual quantum group $U_\hbar\mathfrak{g}$. So there is no really quasi-quantum group attached to a simple finite-dimensional Lie algebra. But, how about the restricted case? That is, do we have quasi-Hopf analogue of Lusztig's definition of a small quantum group?

The aim of the paper and following works is to find the quasi-Hopf analogues of Lusztig's small quantum groups and consequently give some new examples of finite dimensional nonsemisimple quasitriangular quasi-Hopf algebras. As a try, we want to give the quasi-Hopf analogue of $\mathbf{u}_q(\mathfrak{sl}_2)$ in this paper. Inspired by the Hopf case, one can believe that it should be the Drinfeld double of the quasi-Hopf analogue of a Taft algebra. Meanwhile, the general theory of the Drinfeld double for a quasi-Hopf algebra was already developed by Majid, Hausser-Nill and Schauenburg [10, 6, 11] and the quasi-Hopf analogues, denoted by $A(n, q)$, of Taft algebras were discovered by Gelaki [4]. So all things were prepared, and the only task is to compute them out.

But, before computation, Etingof and Gelaki shew that almost nothing new will be created [5]! Precisely, as one conclusion of the main result in [5], they proved that the double $D(A(n, q))$ is always twist equivalent to $\mathbf{u}_q(\mathfrak{sl}_2)$ if n is odd. There is a restriction on Etingof-Gelaki's result, that is, n must be odd. Is this condition necessary? Our answer is YES. As one of main results of this paper, we show that $D(A(n, q))$ is *not* twist equivalent to a Hopf algebra if n is even and consequently

the quasi-Hopf analogue of $\mathbf{u}_q(\mathfrak{sl}_2)$ is gotten. We will prove the result in a general setting.

In [7], all pointed Majid algebras $M(n, s, q)$ of finite representation type are classified. Such pointed Majid algebras are indeed the dual of the class of basic quasi-Hopf algebras $A(n, s, q)$ which can be considered as the quasi-Hopf analogues of generalized Taft algebras. Note that the quasi-Hopf algebras $A(n, s, q)$ also appeared in [1]. Maybe, the only contribution of this paper is to compute $D(A(n, s, q))$ out explicitly. The main result of paper can be described as follows.

Theorem 1.1. (1) *As a quasi-Hopf algebra, $D(A(n, s, q)) \cong Q_s \mathbf{u}_q(\mathfrak{sl}_2)$.*

(2) *Assume that $n = 2^m l$ and $s = 2^{m'} l'$ with $(l, 2) = (l', 2) = 1$. If $m' < m$, then $D(A(n, s, q))$ is not twist equivalent to a Hopf algebra.*

The quasi-Hopf algebra $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ will be described in Section 2 by using generators and relations. All other preliminaries are also collected in this section. The first part of Theorem 1.1 will be proved in Section 3 and the method is computation. The proof of the second part will be given in Section 4. The main idea of this section is to find some suitable representations of $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ such that they form a subtensor category of $\text{Rep-}Q_s \mathbf{u}_q(\mathfrak{sl}_2)$. By using group cohomologies, we will find that the restriction of the reassociator to this subtensor category is not trivial.

Throughout, we work over an algebraically closed field κ of characteristic 0 and $[\frac{a}{b}]$ denote the Gaussian fraction function. That is, for any natural numbers a, b , $[\frac{a}{b}]$ denotes the biggest integer which is not bigger than $\frac{a}{b}$. About general background knowledge, the reader is referred to [3] for quasi-Hopf algebras, to [2, 8] for general theory about tensor categories, and to [7] for pointed Majid algebras.

2. PRELIMINARIES

In this section we recall the constructions of quasi-Hopf analogues of (generalized) Taft algebras, their dualities and the Drinfeld double of a quasi-Hopf algebra for the convenience of the reader. At last, we will introduce a new quasi-Hopf algebra $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$.

2.1. Path coalgebras and pointed Majid algebra $M(n, s, q)$. The main aim of this subsection is to recall the definition of the pointed Majid algebra $M(n, s, q)$ constructed in [7]. To attack it, the concept path coalgebra is needed.

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t : Q_1 \rightarrow Q_0$ are two maps assigning respectively the source and the target for each arrow. A path of length $l \geq 1$ in the quiver Q is a finitely ordered sequence of l arrows $a_l \cdots a_1$ such that $s(a_{i+1}) = t(a_i)$ for $1 \leq i \leq l - 1$. By convention a vertex is said to be a trivial path of length 0. For a quiver Q , the associated path coalgebra κQ is the κ -space spanned by the set of paths with counit and comultiplication maps defined by $\varepsilon(g) = 1$, $\Delta(g) = g \otimes g$ for each $g \in Q_0$, and for each nontrivial path $p = a_n \cdots a_1$, $\varepsilon(p) = 0$,

$$\Delta(a_n \cdots a_1) = p \otimes s(a_1) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(a_n) \otimes p.$$

The length of paths gives a natural gradation to the path coalgebra. Let Q_n denote the set of paths of length n in Q , then $\kappa Q = \bigoplus_{n \geq 0} \kappa Q_n$ and $\Delta(Q_n) \subseteq \bigoplus_{n=i+j} Q_i \otimes Q_j$. Clearly κQ is pointed with the set of group-likes $G(Q) = Q_0$, and has the following coradical filtration

$$\kappa Q_0 \subseteq \kappa Q_0 \oplus \kappa Q_1 \subseteq \kappa Q_0 \oplus \kappa Q_1 \oplus \kappa Q_2 \subseteq \cdots.$$

Hence κQ is coradically graded.

A dual quasi-bialgebra, or Majid bialgebra for short, is a coalgebra (M, Δ, ε) equipped with a compatible quasi-algebra structure. Namely, there exist two coalgebra homomorphisms

$$M : H \otimes H \rightarrow H, \quad a \otimes b \mapsto ab, \quad \mu : \kappa \rightarrow H, \quad \lambda \mapsto \lambda 1_H$$

and a convolution-invertible map $\Phi : H^{\otimes 3} \rightarrow k$ called reassociator, such that for all $a, b, c, d \in H$ the following equalities hold:

$$(2.1) \quad a_{(1)}(b_{(1)}c_{(1)})\Phi(a_{(2)}, b_{(2)}, c_{(2)}) = \Phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)}b_{(2)})c_{(2)},$$

$$(2.2) \quad 1_H a = a = a 1_H,$$

$$(2.3) \quad \begin{aligned} & \Phi(a_{(1)}, b_{(1)}, c_{(1)}d_{(1)})\Phi(a_{(2)}b_{(2)}, c_{(2)}, d_{(2)}) \\ &= \Phi(b_{(1)}, c_{(1)}, d_{(1)})\Phi(a_{(1)}, b_{(2)}c_{(2)}, d_{(2)})\Phi(a_{(3)}, b_{(1)}, c_{(3)}), \end{aligned}$$

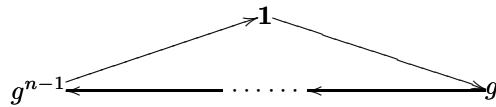
$$(2.4) \quad \Phi(a, 1_H, b) = \varepsilon(a)\varepsilon(b).$$

Here and below we use the Sweedler sigma notation $\Delta(a) = a_{(1)} \otimes a_{(2)} = a' \otimes a''$ for the comultiplication and $a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$ for the result of the n -iterated application of Δ on a . H is called a Majid algebra if, moreover, there exist a coalgebra antimorphism $S : H \rightarrow H$ and two functionals $\alpha, \beta : H \rightarrow \kappa$ such that for all $a \in H$,

$$(2.5) \quad S(a_{(1)})\alpha(a_{(2)})a_{(3)} = \alpha(a)1_H, \quad a_{(1)}\beta(a_{(2)})S(a_{(3)}) = \beta(a)1_H,$$

$$(2.6) \quad \begin{aligned} & \Phi(a_{(1)}, S(a_{(3)}), a_{(5)})\beta(a_{(2)})\alpha(a_{(4)}) = \\ & \Phi^{-1}(S(a_{(1)}), a_{(3)}, S(a_{(5)}))\alpha(a_{(2)})\beta(a_{(4)}) = \varepsilon(a). \end{aligned}$$

A Majid algebra H is said to be pointed, if the underlying coalgebra is pointed. Now we consider a very simple quiver and hope to build a pointed Majid algebra structure on its path coalgebra. The quiver being considered is the following one.



As in [7], this quiver is denoted by $Q(\mathbb{Z}_n, g)$. Now let $0 \leq s \leq n-1$ be a natural number which is a factor of n , i.e., $s|n$, q an n^2 -th primitive root of unity and $\mathfrak{q} := q^n$. Let p_i^l denote the path in $Q(\mathbb{Z}_n, g)$ starting from g^i with length l . So $p_i^0 = g^i$. Let Φ_s be the 3-cocycle over \mathbb{Z}_n defined by

$$(2.7) \quad \Phi_s(g^i, g^j, g^k) = \mathfrak{q}^{si[\frac{i+j}{n}]}, \quad 0 \leq i, j, k \leq n-1.$$

To define $M(n, s, q)$, the definition of the Gaussian binomial coefficient is needed. For any $h \in \kappa$, define $l_h = 1 + h + \cdots + h^{l-1}$ and $l!_h = 1_h \cdots l_h$. The Gaussian binomial coefficient is defined by $\binom{l+m}{l}_h := \frac{(l+m)!_h}{l!_h m!_h}$.

Now we can define the pointed Majid algebra $M(n, s, q)$. As a coalgebra, $M(n, s, q) = \oplus_{i < n[\frac{n}{s}]} \kappa Q(\mathbb{Z}_n, g)_i$. The reassociator, the multiplication, the functions α, β and the antipode are given through

$$(2.8) \quad \Phi(p_i^l, p_j^m, p_k^t) = \delta_{lmt,0} \Phi_s(g^i, g^j, g^k),$$

$$(2.9) \quad p_i^l \cdot p_j^m = q^{-sjl} \mathbf{q}^{s(i+l)[\frac{m+j}{n}]} \binom{l+m}{l}_{q^{-s}} p_{i+j}^{l+m},$$

$$(2.10) \quad \alpha(p_i^l) = \delta_{l,0} \frac{1}{\Phi_s(g^i, g^{n-i}, g^i)}, \quad \beta(p_i^l) = \delta_{l,0} 1,$$

$$(2.11) \quad S(g^i) = g^{n-i}, \quad S(p_0^1) = \mathbf{q}^{-s} p_{n-1}^1,$$

for $0 \leq l, m, t < \frac{n^2}{s}$ and $0 \leq i, j, k \leq n-1$, where $\delta_{a,b}$ is the Kroneck notation which equals to 1 if $a = b$ and 0 otherwise.

Remark 2.1. To get simplicity, we change the multiplication formula defined in Corollary 3.9 of [7] slightly into our formula (2.9). To recover the original formula given in Corollary 3.9 of [7] from (2.9), just substitute q by $\mathbf{q}q$.

2.2. Quasi-Hopf algebra $A(n, s, q)$. A quasi-bialgebra $(H, M, \mu, \Delta, \varepsilon, \phi)$ is a κ -algebra (H, M, μ) with algebra morphisms $\Delta : H \rightarrow H \otimes H$ (the comultiplication) and $\varepsilon : H \rightarrow \kappa$ (the counit), and an invertible element $\phi \in H \otimes H \otimes H$ (the reassociator), such that

$$(2.12) \quad (id \otimes \Delta)\Delta(a)\phi = \phi(\Delta \otimes id)\Delta(a), \quad a \in H,$$

$$(2.13) \quad (id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1),$$

$$(2.14) \quad (\varepsilon \otimes id)\Delta = id = (id \otimes \varepsilon)\Delta,$$

$$(2.15) \quad (id \otimes \varepsilon \otimes id)(\phi) = 1 \otimes 1.$$

We denote $\phi = \sum X^i \otimes Y^i \otimes Z^i$ and $\phi^{-1} = \sum \bar{X}^i \otimes \bar{Y}^i \otimes \bar{Z}^i$. Then a quasi-bialgebra H is called a quasi-Hopf algebra if there is a linear algebra antimorphism $S : H \rightarrow H$ (the antipode) and elements $\alpha, \beta \in H$ satisfying for all $a \in H$,

$$(2.16) \quad \sum S(a_{(1)})\alpha a_{(2)} = \alpha \varepsilon(a), \quad \sum a_{(1)}\beta S(a_{(2)}) = \beta \varepsilon(a),$$

$$(2.17) \quad \sum X^i \beta S(Y^i) \alpha Z^i = 1 = \sum S(\bar{X}^i) \alpha \bar{Y}^i \beta S(\bar{Z}^i).$$

We call an invertible element $J \in H \otimes H$ is a *twist* of H if it satisfies $(\varepsilon \otimes id)(J) = (id \otimes \varepsilon)(J) = 1$. For a twist $J = \sum f_i \otimes g_i$ with inverse $J^{-1} = \sum \bar{f}_i \otimes \bar{g}_i$, set

$$\alpha_J := \sum S(\bar{f}_i) \alpha \bar{g}_i, \quad \beta_J := \sum f_i \beta S(g_i).$$

It is explained that given a twist J of H , if β_J is invertible then one can construct a new quasi-Hopf algebra structure $H_J = (H, \Delta_J, \varepsilon, \Phi_J, S_J, \beta_J \alpha_J)$ on the algebra H , where

$$\Delta_J(a) = J \Delta(a) J^{-1}, \quad a \in H,$$

$$\Phi_J = (1 \otimes J)(id \otimes \Delta)(J)(\Delta \otimes id)(J^{-1})(J \otimes 1)^{-1}$$

and

$$S_J(a) = \beta_J S(a) \beta_J^{-1}, \quad a \in H.$$

Next we will give the definition of the quasi-Hopf algebras $A(n, s, q)$, which will include the quasi-Hopf algebras $A(q)$ constructed by Gelaki [4] as special examples. The dualities of such quasi-Hopf algebras were constructed in [7] and will be recalled in the next subsection.

Let n be a positive integer, \mathfrak{q} an n -th primitive root of unity and $\kappa\mathbb{Z}_n$ the cyclic group algebra of order n . We denote a generator of \mathbb{Z}_n by g_2 and define

$$(2.18) \quad 1_i := \frac{1}{n} \sum_{j=0}^{n-1} (\mathfrak{q}^{n-i})^j g_2^j.$$

For any $0 \leq s \leq n-1$ which is a factor of n , i.e., $s|n$, and q an n -th primitive root of \mathfrak{q} , the quasi-Hopf algebra $A(n, s, q)$ is defined as follows. As an associative algebra, it is generated by x, g_2 and satisfies the following relations

$$(2.19) \quad g_2^n = 1, \quad x^{\frac{n^2}{s}} = 0, \quad g_2 x g_2^{-1} = \mathfrak{q} x.$$

The reassociator ϕ_s , the comultiplication Δ , the counit ε , the elements α, β and the antipode S are given through

$$(2.20) \quad \phi_s = \sum_{i,j,k=0}^{n-1} \mathfrak{q}^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k,$$

$$(2.21) \quad \Delta(g_2) = g_2 \otimes g_2, \quad \Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i,$$

$$(2.22) \quad \alpha = g_2^{-s}, \quad \beta = 1$$

$$(2.23) \quad S(g_2) = g_2^{-1}, \quad S(x) = -x \sum_{i=0}^{n-1} q^{s(i-n)} 1_i.$$

Lemma 2.2. *The algebra $(A(n, s, q), M, \mu, \Delta, \varepsilon, \phi_s, S, \alpha, \beta)$ is a quasi-Hopf algebra and isomorphic to $(M(n, s, q)^*, \Delta^*, \varepsilon^*, M^*, \mu^*, \Phi_s^*, S^*, \alpha^*, \beta^*)$.*

Proof. One can show this through direct computations. For our purpose, it is better to establish a direct isomorphism between $M(n, s, q)^*$ and $A(n, s, q)$. To attack it, we need give a dual basis of $M(n, s, q)$. Recall $\{p_i^l | 0 \leq i \leq n-1, 0 \leq l < n[\frac{n}{s}]\}$ is a basis of $M(n, s, q)$. Let $\{(p_i^l)^* | 0 \leq i \leq n-1, 0 \leq l < n[\frac{n}{s}]\}$ be the canonical dual basis of $M(n, s, q)$. Define

$$(2.24) \quad \varphi : A(n, s, q) \rightarrow M(n, s, q)^*, \quad 1_i \mapsto (p_i^0)^*, \quad x \mapsto \sum_{j=0}^{n-1} (p_j^1)^*,$$

where $0 \leq i \leq n-1$. It is tedious to show that φ gives the desired isomorphism of quasi-Hopf algebras between $M(n, s, q)^*$ and $A(n, s, q)$. \square

Remark 2.3. (1) Take $s = 1$ and the resulting quasi-Hopf algebra $A(n, 1, q)$ is indeed isomorphic the quasi-Hopf algebra $A(q)$ constructed in [4]. In this paper, $A(q)$ is denoted by $A(n, q)$ for consistence.

(2) By the definitions of Φ_s and ϕ_s defined in (2.7) and (2.20), it is easy to see the assumption that s is factor of n is not restrictive. Thus, throughout of this paper, we always take this assumption.

2.3. Drinfeld double of a quasi-Hopf algebra. The construction of the Drinfeld double of a quasi-Hopf algebra is not, at least, a trivial generalization from the Hopf to quasi-Hopf case. After all, the double of a Hopf algebra H is modelled on $H \otimes H^*$, with H and H^* as subalgebras. But if H is just a quasi-Hopf algebra, then H^* is not an associative algebra, so one is at a loss looking for an associative algebra structure on $H \otimes H^*$ and even expect that the double should be some kind of hybrid object. Majid [10] settled this problem at first. He gave a conceptual way to show that the double $D(H)$ is still a quasi-Hopf algebra. Hausser and Nill [6] gave a computable realization of $D(H)$ on $H \otimes H^*$. A more explicit version was gotten by Schauenburg [11]. Here we will recall the Schauenburg's construction.

Let $(H, M, \mu, \Delta, \varepsilon, \phi, S, \alpha, \beta)$ be a finite dimensional quasi-Hopf algebra. Assume $\phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \phi^{(3)} = \sum X^i \otimes Y^i \otimes Z^i$ and $\phi^{-1} = \phi^{(-1)} \otimes \phi^{(-2)} \otimes \phi^{(-3)} = \sum \bar{X}^i \otimes \bar{Y}^i \otimes \bar{Z}^i$. Define

$$(2.25) \quad \gamma := \sum (S(U^i) \otimes S(T^i))(\alpha \otimes \alpha)(V^i \otimes W^i),$$

$$(2.26) \quad \mathbf{f} := \sum (S \otimes S)(\Delta^{op}(\bar{X}^i)) \cdot \gamma \cdot \Delta(\bar{Y}^i \beta S(\bar{Z}^i)),$$

$$(2.27) \quad \chi := (\phi \otimes 1)(\Delta \otimes id \otimes id)(\phi^{-1}),$$

$$(2.28) \quad \omega := (1 \otimes 1 \otimes 1 \otimes \tau(\mathbf{f}^{-1}))(id \otimes \Delta \otimes S \otimes S)(\chi)(\phi \otimes 1 \otimes 1),$$

where $(1 \otimes \phi^{-1})(id \otimes id \otimes \Delta)(\phi) = \sum T^i \otimes U^i \otimes V^i \otimes W^i$ and τ is the twist, i.e., $\tau(a \otimes b) = b \otimes a$.

As a linear space, $D(H) = H \otimes H^*$ and we write $h \bowtie \psi := h \otimes \psi \in D(H)$. There are two canonical actions, denoted by \rightharpoonup , \leftharpoonup , of H on H^* . By definition, for any $a, b \in H$ and $\psi \in H^*$

$$\rightharpoonup: H \otimes H^* \longrightarrow H^*, \quad (a \rightharpoonup \psi)(b) = \psi(ba),$$

$$\leftharpoonup: H^* \otimes H \longrightarrow H^*, \quad (\psi \leftharpoonup a)(b) = \psi(ab).$$

Define a map $\mathbf{T}: H^* \rightarrow D(H)$ by

$$(2.29) \quad \mathbf{T}(\psi) = \phi_{(2)}^{(1)} \bowtie S(\phi^{(2)})\alpha\phi^{(3)} \rightharpoonup \psi \leftharpoonup \phi_{(1)}^{(1)}.$$

With such preparations, $D(H)$ can be described as the following form (see Theorems 6.3 and 9.3 in [11]):

Theorem 2.4. *Let H be a finite dimensional quasi-Hopf algebra. The quasi-Hopf structure on $D(H) = H \otimes H^*$, which contains H as a subquasi-Hopf algebra through the embedding $h \hookrightarrow h \bowtie \varepsilon$, is determined by:*

(1) *As an associative algebra, it is generated by H and $\mathbf{T}(H^*)$ and multiplication rule is*

$$\begin{aligned} & (g \bowtie \varphi)(h \bowtie \psi) \\ &= gh_{(1)(2)}\omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \psi \leftharpoonup \omega^{(1)})(\omega^{(4)}S(h_{(2)})) \rightharpoonup \varphi \leftharpoonup h_{(1)(1)}\omega^{(2)}, \quad (\star) \end{aligned}$$

and as a quasi-coalgebra, the comultiplication is given by

$$\begin{aligned} \Delta_D(\mathbf{T}(\psi)) &= \tilde{\phi}^{(2)}\mathbf{T}(\psi_{(1)}) \leftarrow \tilde{\phi}^{(1)}\phi^{(-1)}\phi^{(1)} \\ &\otimes \tilde{\phi}^{(3)}\phi^{(-3)}\mathbf{T}(\phi^{(3)} \rightharpoonup \psi_{(2)} \leftarrow \phi^{-2})\phi^{(2)}, \quad (\star\star) \end{aligned}$$

for $g, h \in H$ and $\varphi, \psi \in H^*$, where $\tilde{\phi}$ denote another copy of ϕ .

(2) The reassociator ϕ_D , the counit ε_D , elements α_D, β_D and the antipode S_D are given by

$$(2.30) \quad \phi_D = \phi \bowtie \varepsilon, \varepsilon_D(\mathbf{T}(\psi)) = \psi(\phi^{(1)}S(\phi^{(2)})\alpha\phi^{(3)}),$$

$$(2.31) \quad \alpha_D = \alpha \bowtie \varepsilon, \quad \beta_D = \beta \bowtie \varepsilon,$$

$$(2.32) \quad S_D(\mathbf{T}(\psi)) = \mathbf{f}^{(2)}\mathbf{T}(\mathbf{f}^{(-2)} \rightharpoonup S^{-1}(\psi) \leftarrow \mathbf{f}^{(1)})\mathbf{f}^{(-1)},$$

for $\psi \in H^*$.

Remark 2.5. (1) By formula (\star) , $1 \bowtie \varepsilon$ is the unit element of $D(H)$. Moreover, as a special case of this formula, we also have

$$(2.33) \quad (1 \bowtie \varphi)(h \bowtie \varepsilon) = h_{(1)(2)} \bowtie S(h_{(2)}) \rightharpoonup \varphi \leftarrow h_{(1)(1)},$$

for $h \in H$ and $\varphi \in H^*$.

(2) In the process of our computations, we find that there are some misprints in [11] and [6]. Especially, there are misprints in the expression of the element \mathbf{f} given both in [11] and [6], the element χ given in [11] and the comultiplication formula given in [11]. The correct versions are (2.26), (2.27) and $(\star\star)$.

2.4. The quasi-Hopf algebra $\mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$. The quasi-Hopf algebra $\mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$ is defined as follows. As an associative algebra, it is generated by four elements g_1, g_2, x, y satisfying

$$(2.34) \quad g_1^n = g_2^{2s}, \quad g_2^n = 1, \quad g_1 g_2 = g_2 g_1, \quad x^{\frac{n^2}{s}} = y^{\frac{n^2}{s}} = 0,$$

$$(2.35) \quad g_1 x g_1^{-1} = \mathbf{q}^{-s} q^{2s} x, \quad g_2 x g_2^{-1} = \mathbf{q} x,$$

$$(2.36) \quad g_1 y g_1^{-1} = \mathbf{q}^s q^{-2s} y, \quad g_2 y g_2^{-1} = \mathbf{q}^{-1} y,$$

$$(2.37) \quad yx - q^s xy = 1 - g_1 g_2^s.$$

Define

$$(2.38) \quad 1_i := \frac{1}{n} \sum_{j=0}^{n-1} (\mathbf{q}^{n-i})^j g_2^j.$$

The reassociator ϕ_s , the comultiplication Δ , the counit ε , the elements α, β and the antipode S are given through

$$(2.39) \quad \phi_s = \sum_{i,j,k=0}^{n-1} \mathbb{Q}^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k,$$

$$(2.40) \quad \Delta(g_1) = g_1 \otimes g_1, \quad \Delta(g_2) = g_2 \otimes g_2,$$

$$(2.41) \quad \Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i,$$

$$(2.42) \quad \Delta(y) = y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0,$$

$$(2.43) \quad \varepsilon(g_1) = \varepsilon(g_2) = 1, \quad \varepsilon(x) = \varepsilon(y) = 0,$$

$$(2.44) \quad \alpha = g_2^{-s}, \quad \beta = 1$$

$$(2.45) \quad S(g_1) = g_1^{-1}, \quad S(g_2) = g_2^{-1},$$

$$(2.46) \quad S(x) = -x \sum_{i=0}^{n-1} q^{s(i-n)} 1_i, \quad S(y) = -g_1^{-1} g_2^{-s} y \sum_{i=0}^{n-1} q^{s(n-i)'} 1_i.$$

where for any integer $i \in \mathbb{N}$, we denote by i' the remainder of division of i by n .

Lemma 2.6. $\mathbb{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$ is a quasi-Hopf algebra.

Proof. We will show that $D(A(n, s, q))$ is isomorphic to $\mathbb{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$ and thus $\mathbb{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$ is a quasi-Hopf algebra. Moreover, it is quasitriangular. Of course, one can show the result by direct computations. Here we only check the equality $\Delta(y)\Delta(x) - q^s \Delta(x)\Delta(y) = \Delta(1) - \Delta(g_1)\Delta(g_2^s)$. Indeed,

$$\begin{aligned} \Delta(y)\Delta(x) &= (y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0) \cdot \\ &\quad (1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i) \\ &= y \otimes \sum_{i=1}^{n-1} q^{si} 1_i x + y g_2^s \otimes 1_0 x + y x \otimes 1 + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i x \\ &\quad + g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + g_1 g_2^s \otimes y 1_0 x + g_1 x \otimes y 1_0 \\ &= y \otimes \sum_{i=1}^{n-1} q^{si} 1_i x + y g_2^s \otimes 1_0 x + y x \otimes 1 + g_1 g_2^s \otimes y x \\ &\quad + g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + g_1 x \otimes y 1_0, \end{aligned}$$

and

$$\begin{aligned}
q^s \Delta(x) \Delta(y) &= q^s (1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i) \cdot \\
&\quad (y \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_1 g_2^s \otimes y \sum_{i=1}^{n-1} 1_i + g_1 \otimes y 1_0) \\
&= q^s [y \otimes \sum_{i=1}^{n-1} q^{s(i-1)} 1_i x + g_1 g_2^s \otimes xy \sum_{i=1}^{n-1} 1_i + g_2^s y \otimes q^{s(n-1)} 1_0 x \\
&\quad + g_1 g_2^s \otimes xy 1_0 + xy \otimes 1 + x g_1 g_2^s \otimes y \sum_{i=1}^{n-1} q^{-s(i-1)} 1_i \\
&\quad + x g_1 \otimes q^{-s(n-1)} y 1_0] \\
&= q^s [y \otimes \sum_{i=1}^{n-1} q^{s(i-1)} 1_i x + g_1 g_2^s \otimes xy + q^{-s} y g_2^s \otimes 1_0 x \\
&\quad + xy \otimes 1 + q^{-s} g_1 g_2^s x \otimes y \sum_{i=1}^{n-1} q^{-si} 1_i + q^{-s} g_1 x \otimes y 1_0].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta(y) \Delta(x) - q^s \Delta(x) \Delta(y) &= (yx - q^s xy) \otimes 1 + g_1 g_2^s \otimes (yx - q^s xy) \\
&= (1 - g_1 g_2^s) \otimes 1 + g_1 g_2^s \otimes (1 - g_1 g_2^s) \\
&= 1 \otimes 1 - g_1 g_2^s \otimes g_1 g_2^s \\
&= \Delta(1) - \Delta(g_1) \Delta(g_2^s).
\end{aligned}$$

□

3. THE DRINFELD DOUBLE OF $A(n, s, q)$

The main result of this section is the following result.

Theorem 3.1. *The Drinfeld double $D(A(n, s, q))$ of $A(n, s, q)$ is isomorphic to the quasi-Hopf algebra $\mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$. That is,*

$$D(A(n, s, q)) \cong \mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2).$$

To show it, we need understand $D(A(n, s, q))$ well. Recall that for any integer $i \in \mathbb{N}$, we denote by i' the remainder of division of i by n . The following lemma is useful in our computations.

Lemma 3.2. *For any two natural numbers i, j , we always have*

$$(3.1) \quad \left[\frac{i+j'}{n} \right] = \left[\frac{i+j}{n} \right] - \left[\frac{j}{n} \right].$$

Proof.

$$\begin{aligned}
\left[\frac{i+j'}{n} \right] &= \left[\frac{i+j - \left[\frac{j}{n} \right] n}{n} \right] \\
&= \left[\frac{i+j}{n} \right] - \left[\frac{j}{n} \right].
\end{aligned}$$

□

The formula (3.1) will be used frequently without explanation. Recall the reassociator of $A(n, s, q)$ is defined to be

$$\phi_s = \sum_{i,j,k=0}^{n-1} q^{si[\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k.$$

The next lemma will give the explicit formalism of the elements γ , \mathbf{f} , χ and ω in such case. Throughout this section, ϕ_s is denoted by ϕ for short when this is no confusion.

Lemma 3.3. *For the quasi-Hopf algebra $A(n, s, q)$, we have*

$$(3.2) \quad \gamma = \sum_{j,k=0}^{n-1} q^{s(j+k)[\frac{j+k}{n}] + sk[\frac{n-j}{n}] - s(j+2k)} 1_j \otimes 1_k,$$

$$(3.3) \quad \mathbf{f} = \sum_{j,k=0}^{n-1} q^{s(j+k)[\frac{j+k}{n}] + sk[\frac{n-j}{n}] - sk} 1_j \otimes 1_k,$$

$$(3.4) \quad \chi = \sum_{i_1, i_2, j, k=0}^{n-1} q^{si_1[\frac{i_2+j}{n}] - s(i_1+i_2)[\frac{j+k}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_j \otimes 1_k,$$

and

$$\begin{aligned} \omega = & \sum_{i_1, i_2, i_3, i_4, i_5=0}^{n-1} q^{si_5 - s(\sum_{t=1}^5 i_t)[\frac{i_4+i_5}{n}] + si_1[\frac{i_2+i_3+i_4}{n}] - si_5[\frac{n-i_4}{n}]} \\ & 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}). \end{aligned}$$

Proof. At first, we have

$$\begin{aligned} T^i \otimes U^i \otimes V^i \otimes W^i : &= (1 \otimes \phi^{-1})(id \otimes id \otimes \Delta)(\phi) \\ &= \sum_{i,j,k=0}^{n-1} q^{-si[\frac{j+k}{n}]} 1 \otimes 1_i \otimes 1_j \otimes 1_k. \\ & \quad \sum_{i_1, j_1, k_1, k_2=0}^{n-1} q^{si_1[\frac{j_1+(k_1+k_2)'}{n}]} 1_{i_1} \otimes 1_{j_1} \otimes 1_{k_1} \otimes 1_{k_2} \\ &= \sum_{i_1, i, j, k=0}^{n-1} q^{-si[\frac{j+k}{n}]} q^{si_1[\frac{i+(j+k)'}{n}]} 1_{i_1} \otimes 1_i \otimes 1_j \otimes 1_k, \end{aligned}$$

and

$$\begin{aligned}
\gamma &= (S(U^i) \otimes S(T^i))(\alpha \otimes \alpha)(V^i \otimes W^i) \\
&= \sum_{i_1, i, j, k=0}^{n-1} \mathbb{Q}^{-s i [\frac{j+k}{n}]} \mathbb{Q}^{s i_1 [\frac{i+(j+k)'}{n}]} S(1_i) g_2^{-s} 1_j \otimes S(1_{i_1}) g_2^{-s} 1_k \\
&= \sum_{i_1, i, j, k=0}^{n-1} \mathbb{Q}^{-s i [\frac{j+k}{n}]} \mathbb{Q}^{s i_1 [\frac{i+(j+k)'}{n}]} \mathbb{Q}^{-s(j+k)} S(1_i) 1_j \otimes S(1_{i_1}) 1_k \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{-s(n-j)' [\frac{j+k}{n}]} \mathbb{Q}^{s(n-k)' [\frac{(n-j)'+(j+k)'}{n}]} \mathbb{Q}^{-s(j+k)} 1_j \otimes 1_k \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{s j [\frac{j+k}{n}] - s k [\frac{n+k}{n}] + s k [\frac{n-j}{n}] + s k [\frac{j+k}{n}] - s(j+k)} 1_j \otimes 1_k \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{s(j+k) [\frac{j+k}{n}] + s k [\frac{n-j}{n}] - s(j+2k)} 1_j \otimes 1_k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{f} &= \sum (S \otimes S)(\Delta^{op}(\overline{X}^i)) \cdot \gamma \cdot \Delta(\overline{Y}^i \beta S(\overline{Z}^i)) \\
&= \sum_{i_1, i_2, j_1, k_1=0}^{n-1} \mathbb{Q}^{-s(i_1+i_2)' [\frac{j_1+k_1}{n}]} (S(1_{i_1}) \otimes S(1_{i_2})) \gamma \Delta(1_{j_1} 1_{(n-k_1)'}) \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{-s((n-j)'+(n-k)')' [\frac{(j+k)'+(n-(j+k)')'}{n}]} \gamma 1_j \otimes 1_k \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{s(j+k)} \gamma 1_j \otimes 1_k \\
&= \sum_{j, k=0}^{n-1} \mathbb{Q}^{s(j+k) [\frac{j+k}{n}] + s k [\frac{n-j}{n}] - s k} 1_j \otimes 1_k.
\end{aligned}$$

The computation for χ is easy. Indeed,

$$\begin{aligned}
\chi &= (\phi \otimes 1)(\Delta \otimes id \otimes id)(\phi^{-1}) \\
&= \sum_{i, j, k=0}^{n-1} \mathbb{Q}^{s i [\frac{j+k}{n}]} 1_i \otimes 1_j \otimes 1_k \otimes 1 \\
&\quad \sum_{i_1, i_2, j_1, k_1=0}^{n-1} \mathbb{Q}^{-s(i_1+i_2)' [\frac{j_1+k_1}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_{j_1} \otimes 1_{k_1} \\
&= \sum_{i_1, i_2, j, k=0}^{n-1} \mathbb{Q}^{s i_1 [\frac{i_2+j}{n}] - s(i_1+i_2) [\frac{j+k}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_j \otimes 1_k.
\end{aligned}$$

The desire element ω can be gotten in the following way:

$$\begin{aligned}
\omega &= (1 \otimes 1 \otimes 1 \otimes \tau(\mathbf{f}^{-1}))(id \otimes \Delta \otimes S \otimes S)(\chi)(\phi \otimes 1 \otimes 1) \\
&= \sum_{j,k=0}^{n-1} \mathbf{q}^{sk-s(j+k)[\frac{j+k}{k}]-sk[\frac{n-j}{n}]} 1 \otimes 1 \otimes 1 \otimes 1_k \otimes 1_j \sum_{i_1, \dots, i_5=0}^{n-1} \\
&\quad \mathbf{q}^{si_1[\frac{(i_2+i_3)'+i_4}{n}]-s(i_1+(i_2+i_3)'[\frac{i_4+i_5}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}) \\
&\quad \sum_{i_1, i_2, i_3=0}^{n-1} \mathbf{q}^{si_1[\frac{i_2+i_3}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes 1 \otimes 1 \\
&= \sum_{i_1, \dots, i_5=0}^{n-1} \mathbf{q}^{si_1[\frac{i_2+i_3}{n}]+s(n-i_4)-s(2n-i_4-i_5)[\frac{(n-i_4)'+(n-i_5)'}{n}]-s(n-i_4)[\frac{n-(n-i_5)'}{n}]} \\
&\quad \mathbf{q}^{si_1[\frac{i_2+i_3+i_4}{n}]-si_1[\frac{i_2+i_3}{n}]-s(i_1+i_2+i_3)[\frac{i_4+i_5}{n}]} 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}) \\
&= \sum_{i_1, \dots, i_5=0}^{n-1} \mathbf{q}^{-si_4+s(i_4+i_5)[\frac{(n-i_4)'+(n-i_5)'}{n}]+si_4[\frac{n-i_5}{n}]+si_1[\frac{i_2+i_3+i_4}{n}]-s(i_1+i_2+i_3)[\frac{i_4+i_5}{n}]} \\
&\quad 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}) \\
&= \sum_{i_1, \dots, i_5=0}^{n-1} \mathbf{q}^{-si_4+s(i_4+i_5)[\frac{2n-i_4-i_5}{n}]-si_5[\frac{n-i_4}{n}]+si_1[\frac{i_2+i_3+i_4}{n}]-s(i_1+i_2+i_3)[\frac{i_4+i_5}{n}]} \\
&\quad 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}) \\
&= \sum_{i_1, \dots, i_5=0}^{n-1} \mathbf{q}^{-si_4+s(i_4+i_5)(1-[\frac{i_4+i_5}{n}])-si_5[\frac{n-i_4}{n}]+si_1[\frac{i_2+i_3+i_4}{n}]-s(i_1+i_2+i_3)[\frac{i_4+i_5}{n}]} \\
&\quad 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}) \\
&= \sum_{i_1, i_2, i_3, i_4, i_5=0}^{n-1} \mathbf{q}^{si_5-s(\sum_{t=1}^5 i_t)[\frac{i_4+i_5}{n}]+si_1[\frac{i_2+i_3+i_4}{n}]-si_5[\frac{n-i_4}{n}]} \\
&\quad 1_{i_1} \otimes 1_{i_2} \otimes 1_{i_3} \otimes S(1_{i_4}) \otimes S(1_{i_5}).
\end{aligned}$$

□

Once the element ω is known, the multiplication rules of $D(A(n, s, q))$ can be determined according to formula (\star) .

Proposition 3.4. *In $D(A(n, s, q))$, we have the following relations*

$$(3.5) \quad (g_2 \bowtie \varepsilon)^n = 1 \bowtie \varepsilon, \quad (x \bowtie \varepsilon)^{\frac{n^2}{s}} = 0,$$

$$(3.6) \quad (g_2 \bowtie \varepsilon)(x \bowtie \varepsilon)(g_2 \bowtie \varepsilon)^{-1} = \mathbf{q}(x \bowtie \varepsilon),$$

$$(3.7) \quad (1 \bowtie g)(g_2 \bowtie \varepsilon) = (g_2 \bowtie \varepsilon)(1 \bowtie g), \quad (\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^n = g_2^{2s} \bowtie \varepsilon,$$

$$(3.8) \quad (1 \bowtie p_0^1)^{\frac{n^2}{s}} = 0, \quad (g_2 \bowtie \varepsilon)(1 \bowtie p_0^1)(g_2 \bowtie \varepsilon)^{-1} = \mathbf{q}^{-1}(1 \bowtie p_0^1),$$

$$(3.9) \quad (\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)(x \bowtie \varepsilon)(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^{-1} = \mathbf{q}^{-s} q^{2s}(x \bowtie \varepsilon),$$

$$(3.10) \quad (\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)(1 \bowtie p_0^1)(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^{-1} = \mathbf{q}^s q^{-2s}(1 \bowtie p_0^1),$$

and

$$\begin{aligned} \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right)(x \bowtie \varepsilon) &= q^s(x \bowtie \varepsilon) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1\right) \\ &= (1 \bowtie \varepsilon) - (g_2^s \sum_{i=0}^{n-1} q^{si} 1_i \bowtie g). \end{aligned}$$

Proof. Formulas (3.5) and (3.6) are clear since $A(n, s, q)$ is a subquasi-Hopf algebra of its double. The first part of formula (3.7) and the second part of formula (3.8) are direct consequences of formula (2.33). For the second part of (3.7), one has

$$\begin{aligned} (1 \bowtie g)(1 \bowtie g) &= \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup g \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup g \leftharpoonup \omega^{(2)}) \\ &= \sum_{i_3=0}^{n-1} \mathfrak{q}^{-s[\frac{(n-1)'+(n-1)'}{n}]} 1_{i_3} \bowtie g^2 = g_2^{-s} \bowtie g^2. \end{aligned}$$

Inductively, one has

$$(3.11) \quad (1 \bowtie g)^i = g_2^{-s(i-1)} \bowtie g^i$$

for $1 \leq i \leq n$. Thus $(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^n = (\sum_{i=0}^{n-1} q^{si} 1_i \bowtie \varepsilon)^n (1 \bowtie g)^n = (\sum_{i=0}^{n-1} \mathfrak{q}^{si} 1_i \bowtie \varepsilon)(1 \bowtie g)^n = g_2^s \bowtie \varepsilon$.

Since the multiplication of $M(n, s, q)$ is not associative in general, we need two notions. For any algebra (maybe not associative) A , let $X \in A$. Define

$$X^{\overrightarrow{l}} =: \overbrace{(\cdots (X \cdot X) \cdot X) \cdots}^l, \quad X^{\overleftarrow{l}} =: \overbrace{(\cdots (X \cdot (X \cdot X)) \cdots)}^l.$$

For the first part of (3.8), we have

$$\begin{aligned} (1 \bowtie p_0^1)(1 \bowtie p_0^1) &= \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup p_0^1 \leftharpoonup \omega^{(1)})(\omega^{(4)} \rightharpoonup p_0^1 \leftharpoonup \omega^{(2)}) \\ &= \sum_{i_3=0}^{n-1} \mathfrak{q}^{s[\frac{1+i_3}{n}]} 1_{i_3} \bowtie (p_0^1)^{\overrightarrow{2}} \\ &= \sum_{i_3=0}^{n-1} \mathfrak{q}^{s[\frac{1+i_3}{n}]} 1_{i_3} \bowtie (p_0^1)^{\overleftarrow{2}}. \end{aligned}$$

Inductively, one can prove the following equalities:

$$(3.12) \quad (1 \bowtie p_0^1)^{\overrightarrow{l}} = g_2^{s[\frac{l}{n}]} \sum_{i_3=0}^{n-1} \mathfrak{q}^{s[\frac{1+i_3}{n}] + s[\frac{2+i_3}{n}] + \cdots + s[\frac{l'-1+i_3}{n}]} 1_{i_3} \bowtie (p_0^1)^{\overleftarrow{l}},$$

$$(3.13) \quad (1 \bowtie p_0^1)^{\overleftarrow{l}} = \mathfrak{q}^{sl'[\frac{l}{n}]} g_2^{s[\frac{l}{n}]} \sum_{i_3=0}^{n-1} \mathfrak{q}^{s[\frac{1+i_3}{n}] + s[\frac{2+i_3}{n}] + \cdots + s[\frac{l'-1+i_3}{n}]} 1_{i_3} \bowtie (p_0^1)^{\overrightarrow{l}}.$$

Comparing with Lemma 3.6 in [7], we indeed have $(1 \bowtie p_0^1)^{\overleftarrow{l}} = (1 \bowtie p_0^1)^{\overleftarrow{l}}$ and $(1 \bowtie p_0^1)^{n[\frac{n}{s}]} = 0$. Now let's prove the formula (3.9).

$$\begin{aligned}
& \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right) (x \bowtie \varepsilon) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} q^{si} 1_i x_{(1)(2)} \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup \varepsilon \leftarrow \omega^{(1)}) (\omega^{(4)} S(x_{(2)}) \rightharpoonup g \leftarrow x_{(1)(1)} \omega^{(2)}) \right) \\
& \quad \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie \varepsilon \right) \left(\sum_{j=1}^{n-1} 1_j x \bowtie q^{-(n-1)s} g + 1_0 x \bowtie q^s g \right) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(x \sum_{j=0}^{n-2} q^{s(j+1)} 1_j \bowtie q^{-(n-1)s} g + x 1_{n-1} \bowtie q^s g \right) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(x \sum_{j=0}^{n-2} \mathbb{Q}^{-s} q^{2s} q^{sj} 1_j \bowtie g + x \mathbb{Q}^{-s} q^{2s} q^{(n-1)s} 1_{n-1} \bowtie g \right) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \mathbb{Q}^{-s} q^{2s} (x \bowtie \varepsilon) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \mathbb{Q}^{-s} q^{2s} (x \bowtie \varepsilon).
\end{aligned}$$

For (3.10), note that $(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g)^{-1} = (g_2^s \sum_{i=0}^{n-1} q^{-si} 1_i \bowtie g^{n-1})$.

$$\begin{aligned}
& \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right) (1 \bowtie p_0^1) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} q^{si} 1_i \omega^{(3)} \bowtie (\omega^{(5)} \rightharpoonup p_0^1 \leftarrow \omega^{(1)}) (\omega^{(4)} \rightharpoonup g \leftarrow \omega^{(2)}) \right) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie g \right)^{-1} \\
&= \left(\sum_{i=0}^{n-1} \mathbb{Q}^s q^{si} 1_i \bowtie (p_0^1 g) \right) (g_2^s \sum_{i=0}^{n-1} q^{-si} 1_i \bowtie g^{n-1}) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{Q}^s q^{si} 1_i \bowtie \varepsilon \right) \left(\sum_{i_3=0}^{n-1} \mathbb{Q}^s \mathbb{Q}^{s-s(1+i_3)+s(n-1)[\frac{n+1+i_3}{n}]+s[\frac{2}{n}]} q^{-s(1+i_3)'} 1_{i_3} g_2^s \bowtie g^{n-1} (p_0^1 g) \right) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{Q}^s q^{si} 1_i \bowtie \varepsilon \right) \left(\sum_{i_3=0}^{n-1} \mathbb{Q}^s \mathbb{Q}^{-s(1+i_3)+s[\frac{2}{n}]} q^{-s(1+i_3)} 1_{i_3} g_2^s \bowtie g^{n-1} (p_0^1 g) \right) \\
&= \left(\sum_{i=0}^{n-1} \mathbb{Q}^s q^{si} 1_i \bowtie \varepsilon \right) \left(\sum_{i_3=0}^{n-1} \mathbb{Q}^{s[\frac{2}{n}]} q^{-s(1+i_3)} 1_{i_3} \bowtie g^{n-1} (p_0^1 g) \right) \\
&= \sum_{i=0}^{n-1} \mathbb{Q}^s q^{-s} \mathbb{Q}^{s[\frac{2}{n}]} 1_i \bowtie g^{n-1} (p_0^1 g) \\
&= \sum_{i=0}^{n-1} \mathbb{Q}^s q^{-s} \mathbb{Q}^{s[\frac{2}{n}]} 1_i \bowtie q^{-s} \mathbb{Q}^{-s[\frac{2}{n}]} p_0^1 \\
&= \mathbb{Q}^s q^{-2s} 1 \bowtie p_0^1,
\end{aligned}$$

where the second last equality are gotten from (2.9).

Now the only task is to prove the last formula in this proposition. Note that

$$\begin{aligned} (\Delta \otimes id)\Delta(x) &= 1 \otimes 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes g_2^s \otimes 1_0 x + 1 \otimes \sum_{i=1}^{n-1} 1_i x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i \\ &\quad + g_2^s \otimes 1_0 x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i \otimes \sum_{i=0}^{n-1} q^{-si} 1_i. \end{aligned}$$

By applying above five items $1 \otimes 1 \otimes \sum_{i=1}^{n-1} 1_i x$, $g_2^s \otimes g_2^s \otimes 1_0 x$, $1 \otimes \sum_{i=1}^{n-1} 1_i x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i$, $g_2^s \otimes 1_0 x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i$ and $x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i \otimes \sum_{i=0}^{n-1} q^{-si} 1_i$ into the following formula

$$(1 \bowtie p_0^1)(x \bowtie \varepsilon) = x_{(1)(2)} \bowtie S(x_2) \rightharpoonup p_0^1 \leftarrow x_{(1)(1)},$$

we get

$$(1 \bowtie p_0^1)(x \bowtie \varepsilon) = \sum_{i=1}^{n-1} 1_i x \bowtie p_0^1 + q^s 1_0 x \bowtie p_0^1 - g_2^s \bowtie g + \sum_{i=0}^{n-1} q^{-si} 1_i \bowtie \varepsilon.$$

Multiplying the element $\sum_{i=0}^{n-1} q^{si} 1_i \bowtie \varepsilon$ to above equality, we have

$$\begin{aligned} \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1 \right) (x \bowtie \varepsilon) &= q^s (x \bowtie \varepsilon) \left(\sum_{i=0}^{n-1} q^{si} 1_i \bowtie p_0^1 \right) \\ &\quad + (1 \bowtie \varepsilon) - (g_2^s \sum_{i=0}^{n-1} q^{si} 1_i \bowtie g). \end{aligned}$$

□

Next, let us determine the comultiplication, the counit, the antipode and the elements α, β of $D(A(n, s, q))$. From now on, sometimes, we denote $h \bowtie \varphi \in D(A(n, s, q))$ by $h\varphi$ for short.

Proposition 3.5. *In $D(A(n, s, q))$, we have*

$$(3.14) \quad \Delta(g_2) = g_2 \otimes g_2,$$

$$(3.15) \quad \Delta(x) = 1 \otimes \sum_{i=1}^{n-1} 1_i x + g_2^s \otimes 1_0 x + x \otimes \sum_{i=0}^{n-1} q^{-si} 1_i,$$

$$(3.16) \quad \Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = \sum_{i=0}^{n-1} q^{si} 1_i g \otimes \sum_{i=0}^{n-1} q^{si} 1_i g,$$

$$(3.17) \quad \Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = \sum_{i=0}^{n-1} q^{si} 1_i p_0^1 \otimes \sum_{i=0}^{n-1} q^{si} 1_i + g_2^s \sum_{i=0}^{n-1} q^{si} 1_i g \otimes \sum_{i=0}^{n-2} q^{si} 1_i p_0^1 \\ + \sum_{i=0}^{n-1} q^{si} 1_i g \otimes q^{s(n-1)} 1_{n-1} p_0^1,$$

$$(3.18) \quad \varepsilon(g_2) = 1, \quad \varepsilon(x) = 0,$$

$$(3.19) \quad \varepsilon\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = 1, \quad \varepsilon\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = 0,$$

$$(3.20) \quad S(g_2) = g_2^{-1}, \quad S(x) = -x \sum_{i=0}^{n-1} q^{s(i-n)} 1_i,$$

$$(3.21) \quad S\left(\sum_{i=0}^{n-1} q^{si} 1_i g\right) = \left(\sum_{i=0}^{n-1} q^{si} 1_i g\right)^{-1},$$

$$(3.22) \quad S\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = -q^{(n-1)s} g_2^{-s} \left(\sum_{i=0}^{n-1} q^{si} g\right)^{-1} p_0^1,$$

$$(3.23) \quad \alpha = g_2^{-s} \bowtie \varepsilon, \quad \beta = 1 \bowtie \varepsilon.$$

Proof. Formulas (3.14), (3.15), (3.18) and (3.20) are clear since $D(A(n, s, q))$ contains $A(n, s, q)$ as a subquasi-Hopf algebra. By (2.29), one can verify directly that

$$\mathbf{T}(g) = g_2^s \bowtie g, \quad \mathbf{T}(p_0^1) = 1 \bowtie p_0^1.$$

Using the comultiplication formula $(\star\star)$, we have

$$(3.24) \quad \Delta(\mathbf{T}(g)) = \sum_{i,j=0}^{n-1} q^{s[\frac{i+j}{n}]} (1_i \otimes 1_j) (\mathbf{T}(g) \otimes \mathbf{T}(g)),$$

$$(3.25) \quad \Delta(\mathbf{T}(p_0^1)) = \sum_{i,j=0}^{n-1} q^{s[\frac{i+j}{n}]} 1_i \mathbf{T}(p_0^1) \otimes 1_j \\ + \sum_{i,j=0}^{n-1} q^{s[\frac{i+j}{n}] - si[\frac{1+j}{n}]} 1_i \otimes 1_j \mathbf{T}(p_0^1).$$

From (3.24), we now know that $\sum_{i=0}^{n-1} q^{si} 1_i \mathbf{T}(g)$ is a group-like element. Since $\mathbf{T}(g) = g_2^s \bowtie g$ and g_2 is group-like, $\sum_{i=0}^{n-1} q^{si} 1_i g$ is a group-like element. Therefore, (3.16) is proved. By Δ is an algebra morphism,

$$\Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i p_0^1\right) = \Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i\right) \Delta(p_0^1) = \Delta\left(\sum_{i=0}^{n-1} q^{si} 1_i\right) \Delta(\mathbf{T}(p_0^1)).$$

Using (3.25) directly, one can get the formula (3.17). Once the comultiplication rule is determined, the counit is clear now. Also, by the definition of the Drinfeld double, we know that $\alpha = g_2^{-s} \bowtie \varepsilon$, $\beta = 1 \bowtie \varepsilon$. From this and the comultiplication formulas (3.16) and (3.17), one can verify that (3.21) and (3.22) are the desired formulas for the antipode. \square

Proof of Theorem 3.1. Define a map

$$\begin{aligned} \Psi : Q_s \mathbf{u}_q(\mathfrak{sl}_2) &\rightarrow D(A(n, s, q)), & g_1 &\mapsto \sum_{i=0}^{n-1} q^{si} 1_i g, & g_2 &\mapsto g_2, \\ & & x &\mapsto x, & y &\mapsto \sum_{i=0}^{n-1} q^{si} 1_i p_0^1. \end{aligned}$$

By Proposition 3.4, it is an algebra morphism. It is also surjective by Theorem 2.4 (1). Comparing the dimensions of two algebras, Ψ is a bijection. To show the result, it is enough to show that it is a coalgebra morphism. This is a direct consequence of Proposition 3.5 by noting that in the formula (3.17), $\sum_{i=0}^{n-2} q^{si} 1_i p_0^1 = (\sum_{i=0}^{n-1} q^{si} 1_i p_0^1) \sum_{j=1}^{n-1} 1_j$ and $q^{s(n-1)} 1_{n-1} p_0^1 = (\sum_{i=0}^{n-1} q^{si} 1_i p_0^1) 1_0$. The theorem is proved. \square

4. TWIST EQUIVALENCE

We give a sufficient condition to determine when $D(A(n, s, q))$ is not trivial, i.e., not twist equivalent to a Hopf algebra.

Theorem 4.1. *Assume that $n = 2^m l$ and $s = 2^{m'} l'$ with $(l, 2) = (l', 2) = 1$. If $m' < m$, then $D(A(n, s, q))$ is not twist equivalent to a Hopf algebra.*

Proof. At first, there is no harm to assume that n is divided by s . Secondly, we can identify $D(A(n, s, q))$ with $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ by Theorem 3.1. We construct a 1-dimensional representation for $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ through the following algebra morphism

$$\rho : Q_s \mathbf{u}_q(\mathfrak{sl}_2) \rightarrow \kappa, \quad g_1 \mapsto -1, \quad g_2 \mapsto (-1)^{\frac{1}{s}}, \quad x \mapsto 0, \quad y \mapsto 0.$$

One can check directly that ρ is well-defined. Denote this representation by X . Let $\text{Rep-}Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ be the representation category of $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$. It is a tensor category.

Let $\langle X \rangle$ be the subtensor category generated by X . Explicitly, define

$$X^{\vec{\otimes} i} =: \overbrace{(\cdots (X \otimes X) \otimes X) \cdots}^i.$$

Then the objects of $\langle X \rangle$ are direct sums of elements being in $\{X^{\vec{\otimes} i} | 0 \leq i < 2s\}$. Now assume that $Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ is twist equivalent to a Hopf algebra. By the general principle of Tannak-Krein duality (see, e.g., [2]), there is a fiber functor from the category $\text{Rep-}Q_s \mathbf{u}_q(\mathfrak{sl}_2)$ to the category of κ -spaces. Thus its restriction to $\langle X \rangle$ is still a fiber functor. This implies the restriction of ϕ_s to $\langle X \rangle$ should be gotten from

a 3-coboundary of \mathbb{Z}_{2s} . It is not hard to see that

$$\begin{aligned}\phi_s|_{\langle X \rangle} &= \sum_{i,j,k=0}^{2s} \mathbb{Q}^{s \frac{ni}{2s} [\frac{nj}{2s} + \frac{nk}{2s}]} 1_{\frac{ni}{2s}} \otimes 1_{\frac{nj}{2s}} \otimes 1_{\frac{nk}{2s}} \\ &= \sum_{i,j,k=0}^{2s} (-1)^{i[\frac{j+k}{2s}]} 1_{\frac{ni}{2s}} \otimes 1_{\frac{nj}{2s}} \otimes 1_{\frac{nk}{2s}}.\end{aligned}$$

By the general theory of group cohomology, it is known that the 3-cocycle

$$f(g_{2s}^i, g_{2s}^j, g_{2s}^k) = (-1)^{i[\frac{j+k}{2s}]}, \quad 0 \leq i, j, k < 2s$$

is not a 3-coboundary where g_{2s} denoting a generator of \mathbb{Z}_{2s} . That's a contradiction. \square

Corollary 4.2. *If n is even and s is odd, then $D(A(n, s, q))$ is not twist equivalent to a Hopf algebra.*

Corollary 4.3. *The quasitriangular quasi-Hopf algebra $D(A(n, q))$ is twist equivalent to $\mathbf{u}_q(\mathfrak{sl}_2)$ if and only if n is odd.*

Proof. As pointed out in Remark 2.3, $A(n, q) = A(n, 1, q)$ and thus the “only if” part is just a direct consequence of Corollary 4.2. The sufficiency is prove in [5] by using conceptual way. Here we give another proof. Let $n = 2m + 1$ and construct $1_i^{n^2} := \frac{1}{n^2} \sum_{j=0}^{n^2-1} q^{-ij} (g_1^{m+1})^{ij}$. Define

$$J := \sum_{i,j=0}^{n^2-1} q^{i(j-j')} 1_i^{n^2} \otimes 1_j^{n^2}.$$

One can verify that $\phi_{J^{-1}} = (1 \otimes J^{-1})(id \otimes \Delta)(J^{-1})\phi_1(\Delta \otimes id)(J)(J \otimes 1) = 1 \otimes 1 \otimes 1$. Thus $D(A(n, q))_{J^{-1}}$ is a Hopf algebra, which is $\mathbf{u}_q(\mathfrak{sl}_2)$ obviously. \square

Remark 4.4. (1) In particular, the quasitriangular quasi-Hopf algebra $\mathbf{Q}_1 \mathbf{u}_q(\mathfrak{sl}_2)$ is not twist equivalent to a Hopf algebra if n is even. In this case, we denote it by $\mathbf{Q} \mathbf{u}_q(\mathfrak{sl}_2)$. Its representation theory will be given elsewhere.

(2) The subtensor category $\langle X \rangle$ constructed in the proof of Theorem 4.1 can be realized as the representation category of the following quasi-Hopf algebra. Let χ be the character of X and define $I := \bigcap_{i=0}^{2s} \text{Ker } \chi^i$. I is a Hopf ideal of $\mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)$. Then $\langle X \rangle$ is isomorphic to $\text{Rep-}\mathbf{Q}_s \mathbf{u}_q(\mathfrak{sl}_2)/I$.

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